Metric behavior of generalizations of Thompson's group F

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Groups of the Form $F(n_1, \ldots, n_k)$ have been considered by Bieri, Strebel, and Stein, but the metric properties of these groups have not yet been considered.

- We consider $F(n_1, \ldots, n_k)$ for
  - $n_1, \ldots, n_k \in \{2, 3, 4, \ldots, \}$,
  - $k \in \{2, 3, 4, \ldots\}$,
  - where $n_1 - 1 | n_j - 1$ for all $j \in \{1, \ldots, k\}$
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  - and we assume $n_1, ..., n_k$ are relatively prime.
- $F(n_1, ..., n_k)$ is the group of piecewise-linear orientation-preserving homeomorphisms of the closed unit interval with finitely-many breakpoints in $\mathbb{Z}[\frac{1}{n_1 n_2 \cdots n_k}]$ and slopes in the cyclic multiplicative group $\langle n_1, n_2, ..., n_k \rangle$ in each linear piece.

Tree-pair diagram representatives

As in the case of F, any element of $F(n_1, ..., n_k)$ can be represented using a $(n_1, ..., n_k)$–ary tree-pair diagram:

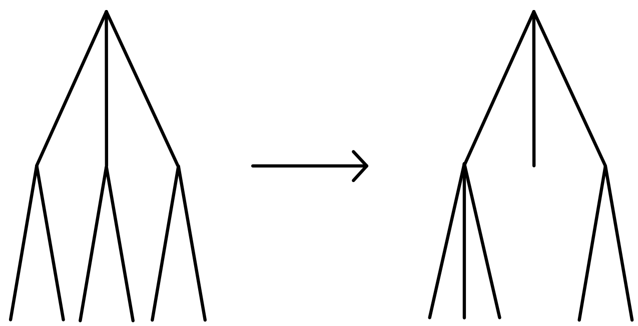

Figure: A $(2, 3)$–ary tree-pair diagram.

Composition

Composition of $(n_1, ..., n_k)$–ary tree-pair diagrams:

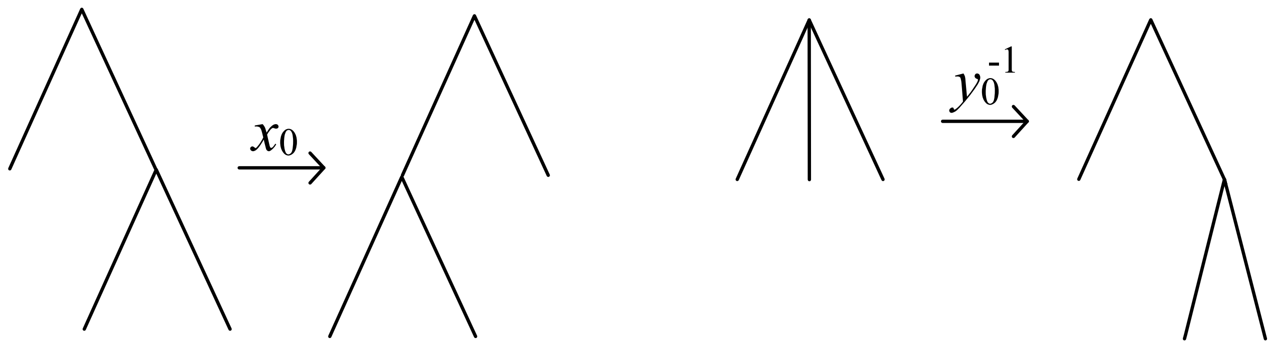

Figure: Composition of two sample elements in $F(2,3)$

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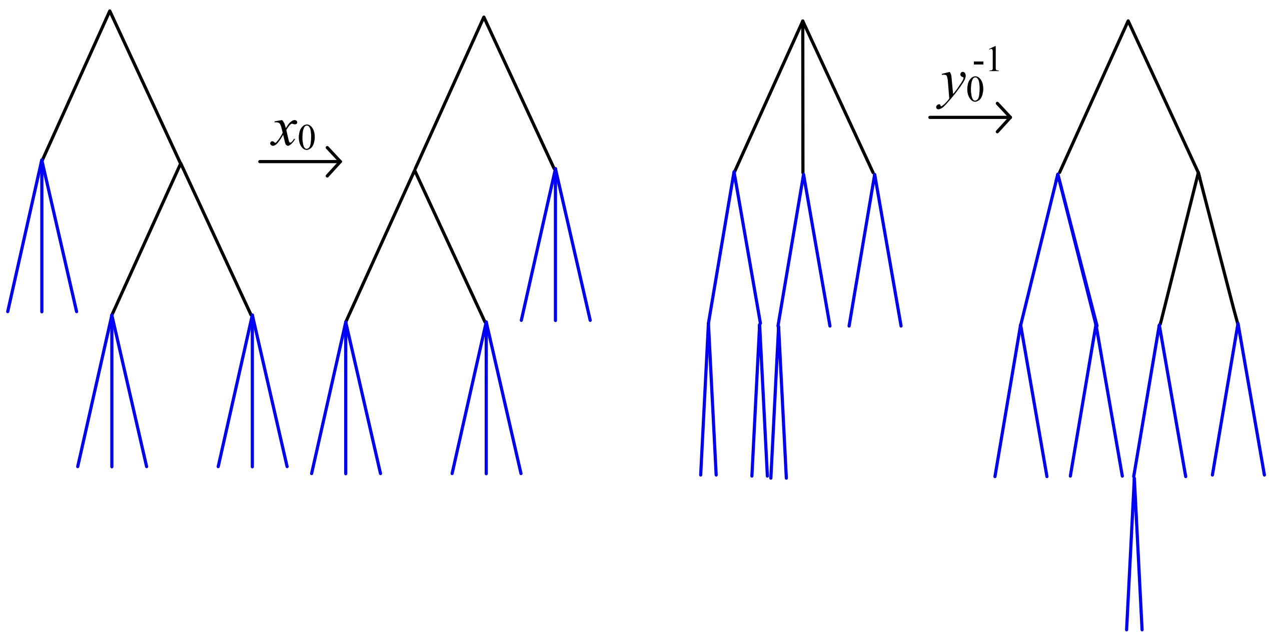

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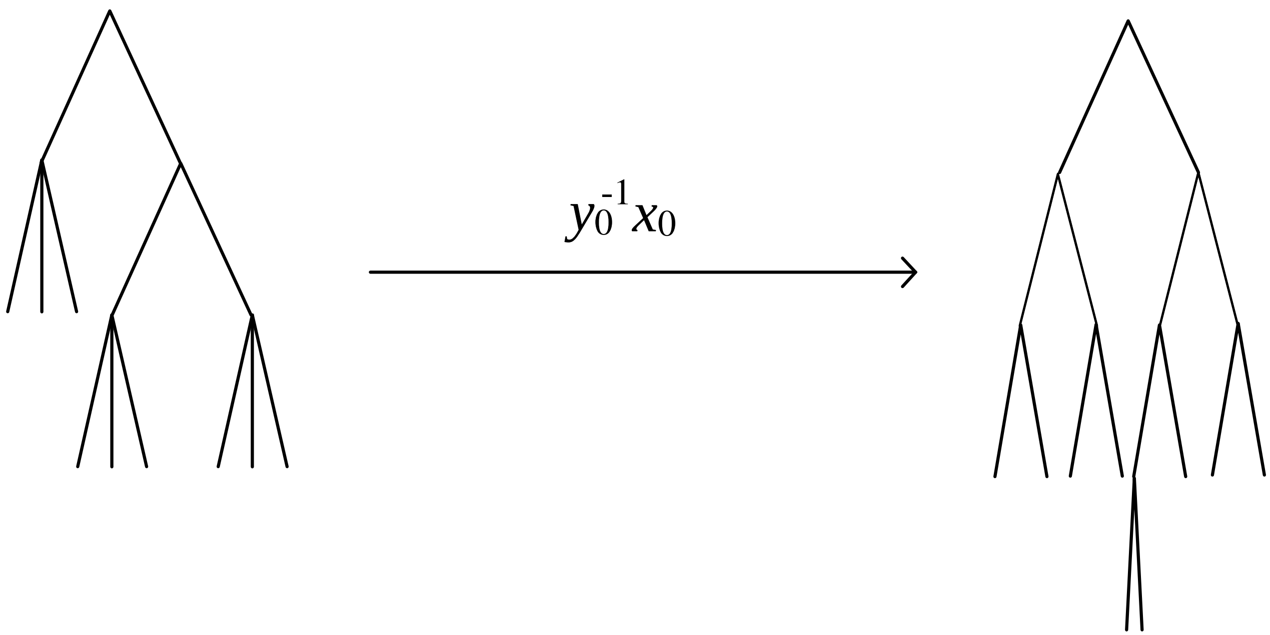

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Minimal Tree-Pair Diagrams

Definition

An $(n_1, ..., n_k)$–ary tree-pair diagram is *minimal* if it has the minimal possible number of leaves of all tree-pair diagram representatives of the element of $F(n_1, ..., n_k)$ which it represents.

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Non-minimal tree-pair diagrams may not contain an exposed caret pair:

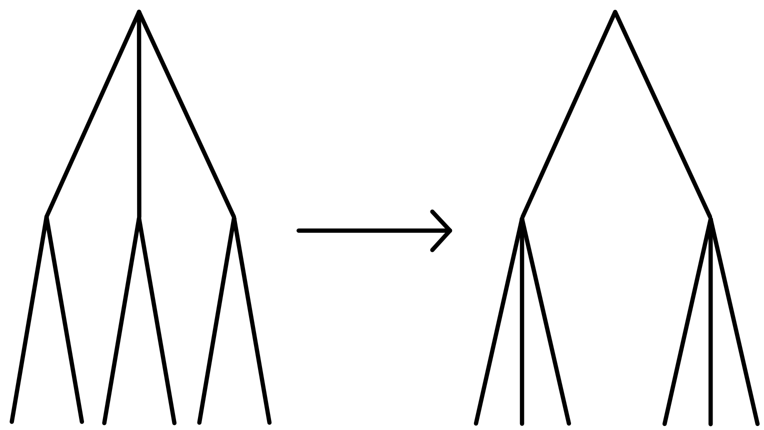
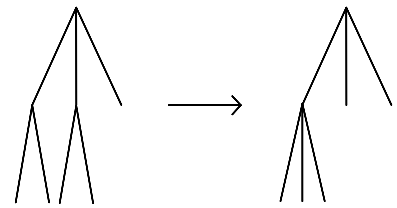


Figure: Two equivalent but distinct minimal tree-pair diagrams representing an element of $F(2, 3)$

Minimal Tree-Pair Diagrams

We may need to add caret pairs to a tree-pair diagram representative in order to obtain a minimal tree-pair diagram:



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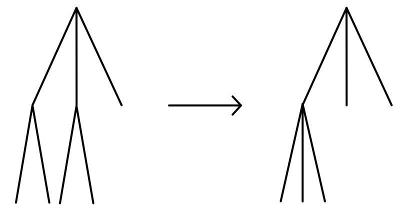

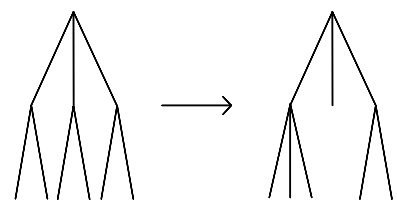

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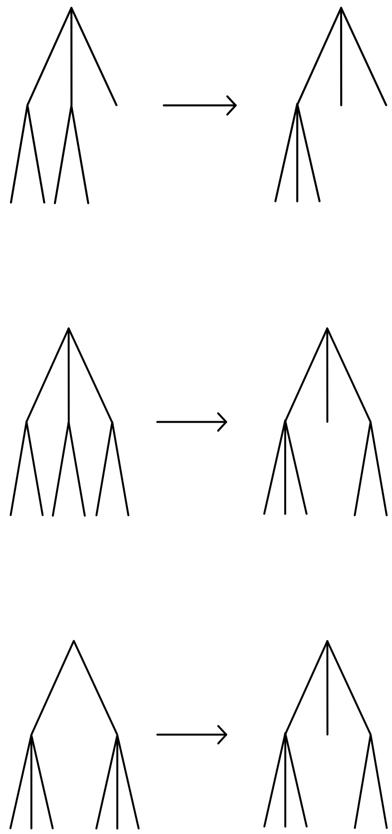

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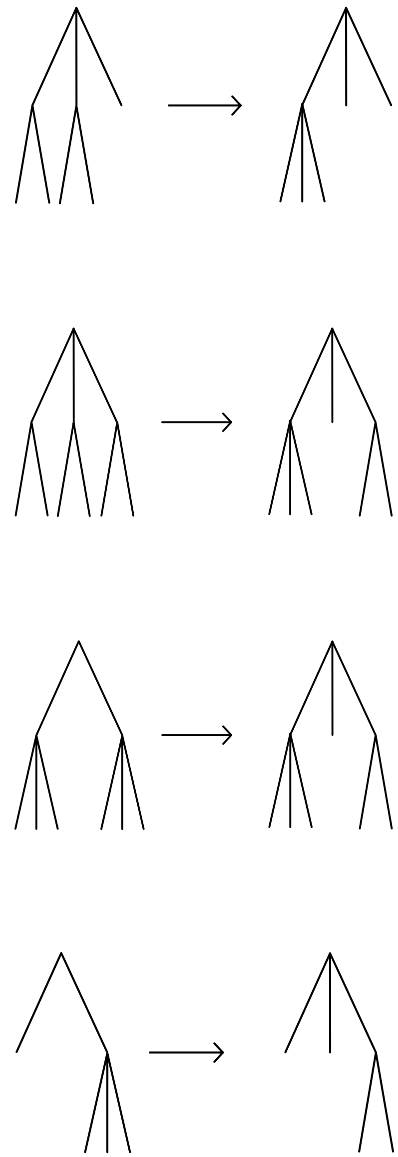


Minimal Tree-Pair Diagrams

Minimal tree-pair diagram representatives of an element of $F(n_1, ..., n_k)$ may not be unique:

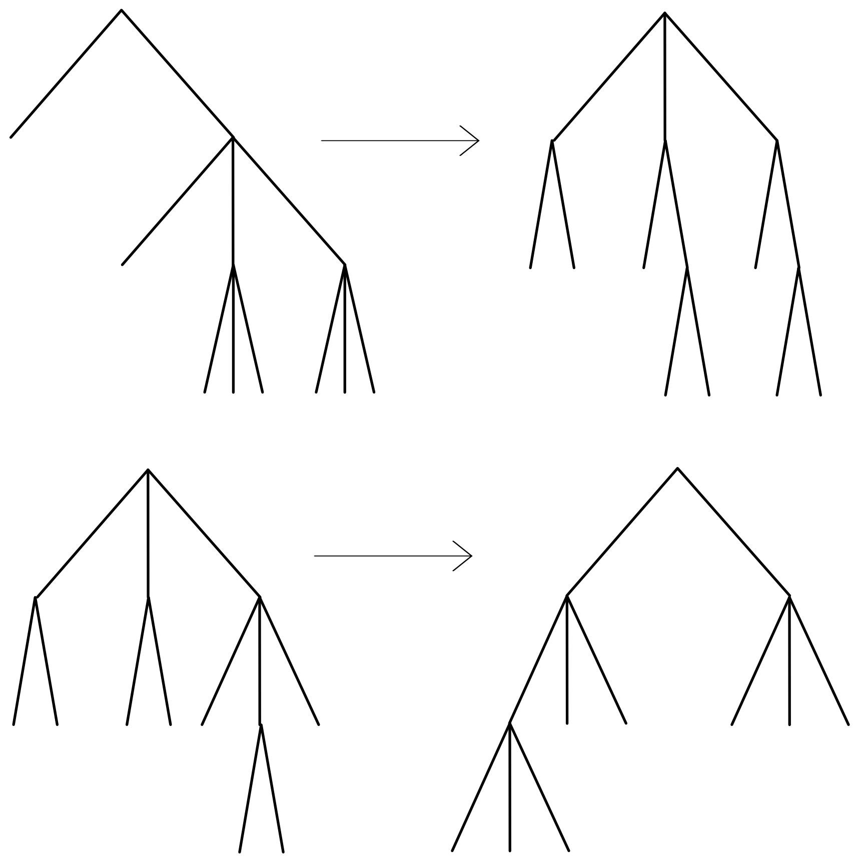


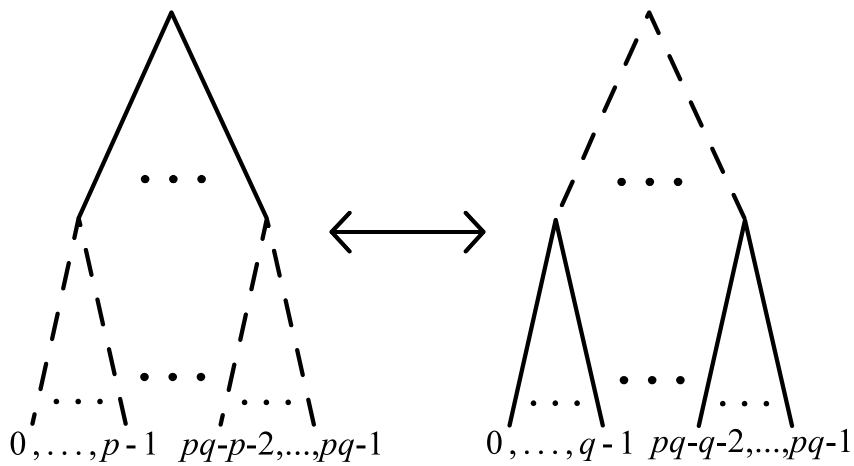
Figure: Two equivalent minimal tree-pair diagrams

Minimal Tree-Pair Diagrams

Theorem

An $(n_1, \ldots, n_k)$–ary tree-pair diagram can be obtained from any equivalent tree-pair diagram by a finite sequence of:

- *Removal of exposed caret pairs*
- *Subtree substitutions of the following type (where $p, q \in \{n_1, \ldots, n_k\}, p \neq q$):*



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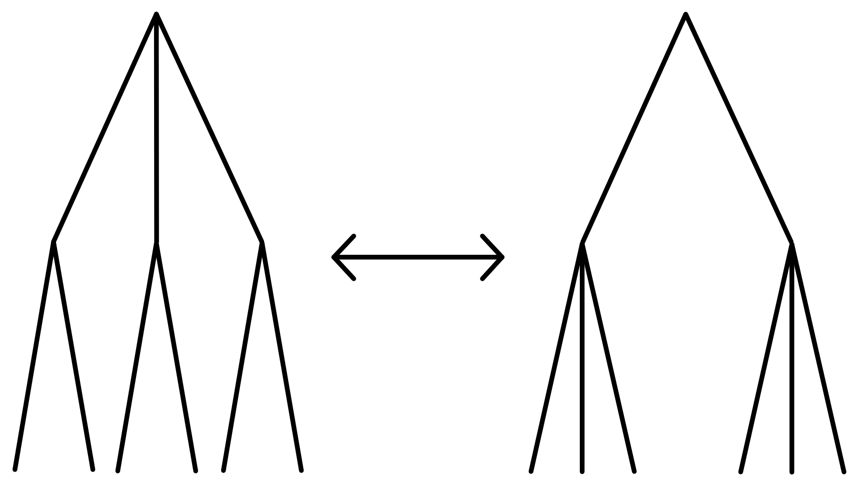
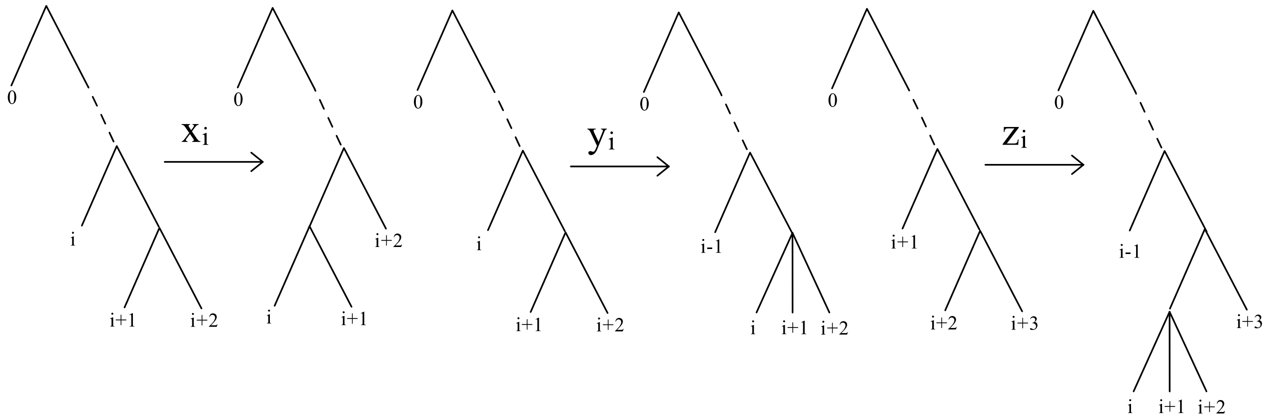


Infinite Presentation (Stein)

Generators:

$$\{x_0, x_1, \ldots, y_0, y_1, \ldots, z_0, z_1, \ldots\}$$



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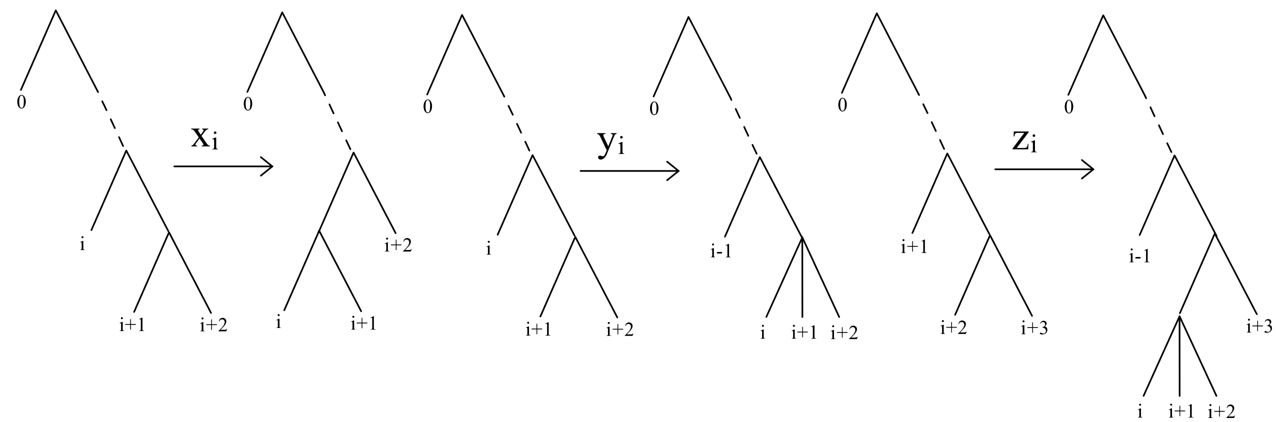

Relators:

1. $x_j x_i = x_i x_{j+1}$
2. $y_j x_i = x_i y_{j+1}$
3. $z_j x_i = x_i z_{j+1}$
4. $x_j z_i = z_i x_{j+2}$
5. $y_j z_i = z_i y_{j+2}$
6. $z_j z_i = z_i z_{j+2}$

for $i < j$ and

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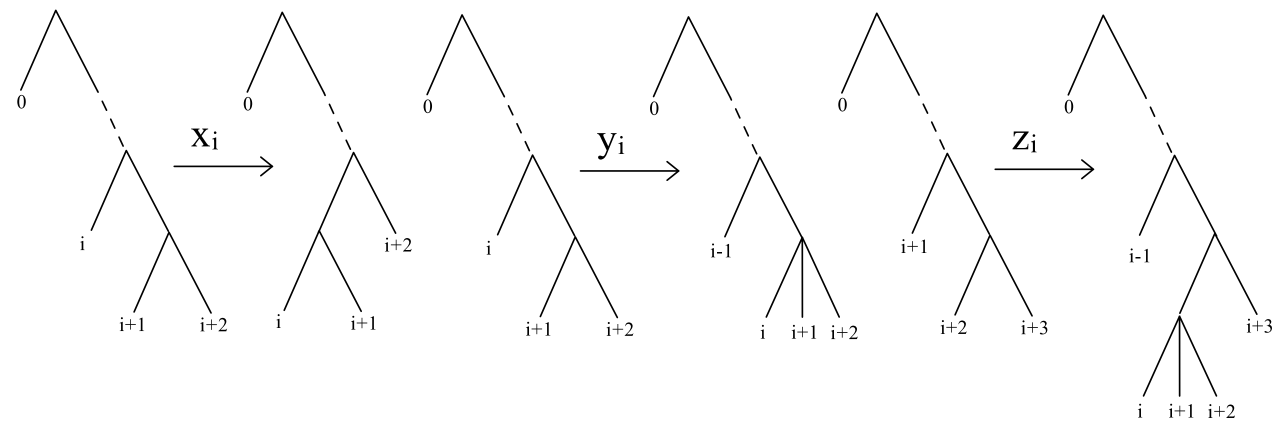
$$\{x_0, x_1, ..., y_0, y_1, ..., z_0, z_1, ...\}$$



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for $i < j$ and

1. $y_{i+1} z_i = y_i x_{i+1} x_i$
2. $x_i z_{i+1} z_i = z_i x_{i+2} x_{i+1} x_i$

for all i.

Normal Form

- We can obtain a unique minimal tree-pair diagram representative for a given equivalence class of tree-pair diagram representatives if we impose a specific set of conditions for choosing among the set of minimal tree-pair diagrams of that class.

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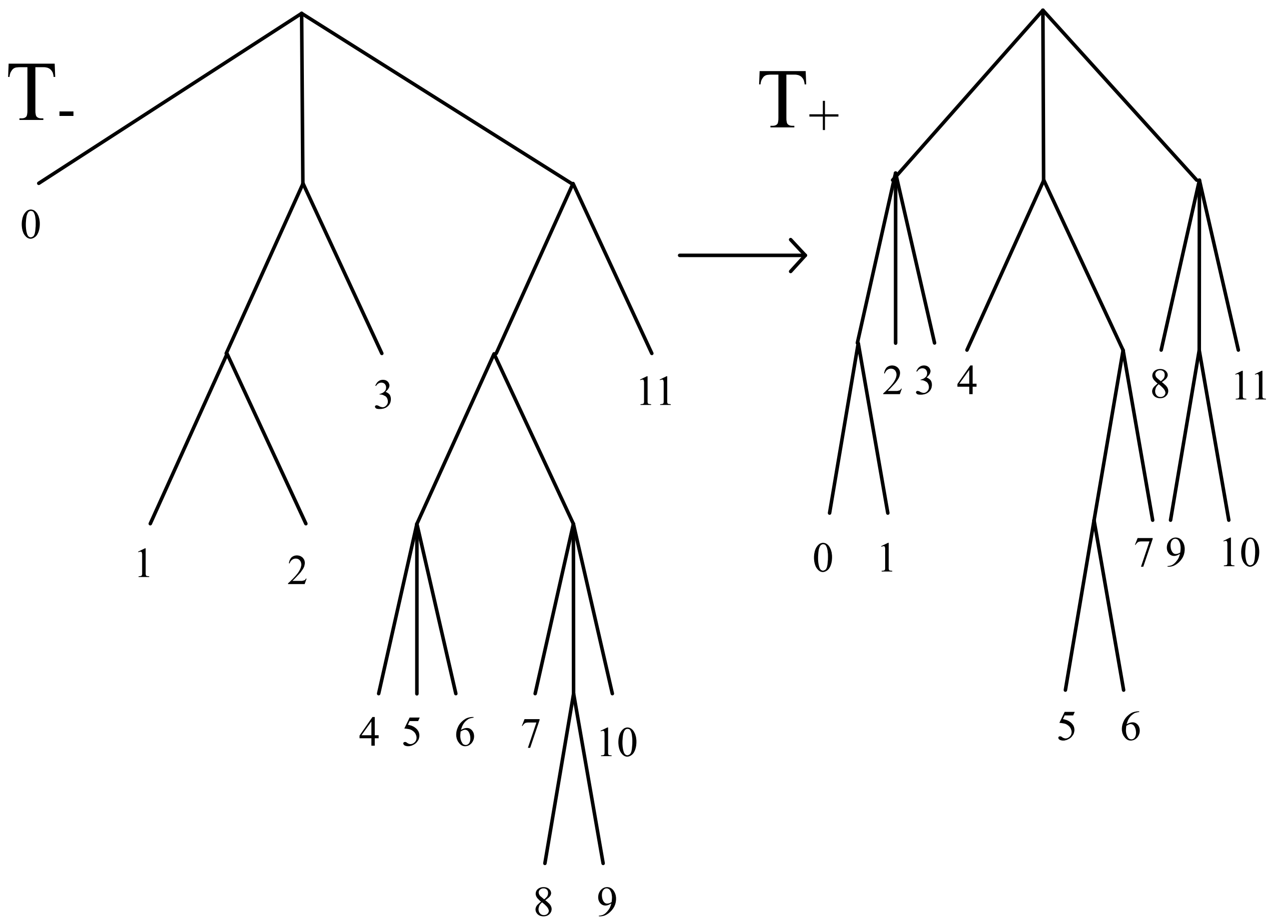
- Using a technique similar to that in F, we can then obtain a unique algebraic expression (the *normal form*) for any element of $F(n_1, ..., n_k)$ by using its unique minimal tree-pair diagram representative.

Normal Form

An example calculating the normal form of an element of $F(2,3)$:



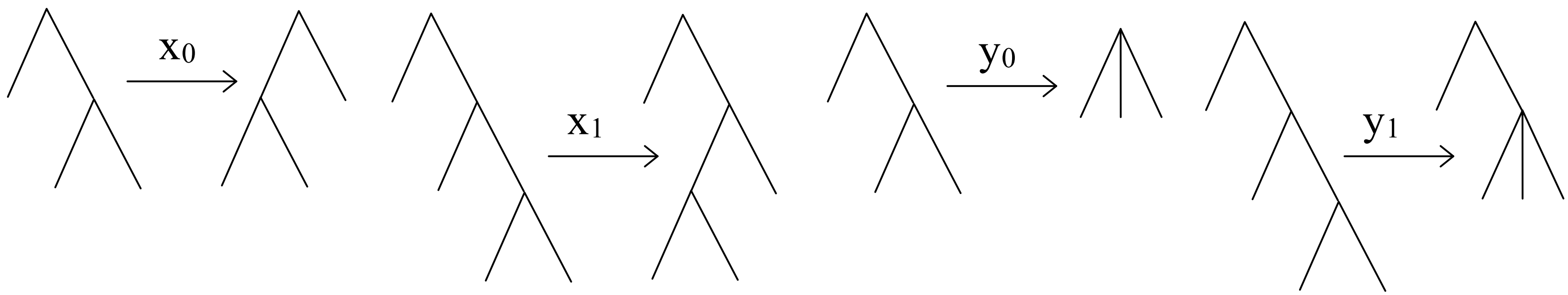
The normal form of the element of $F(2,3)$ represented by this tree-pair diagram is:

$$y_0 y_2 z_0 x_0 x_4 x_5^2 x_9 (y_0 x_1^2 x_4 z_4 z_7 x_8)^{-1}$$

Finite Presentation (Stein)

Generators:

$$\{x_0, x_1, y_0, y_1\}$$



Relators:

1. $x_2x_0 = x_0x_3$
2. $x_3x_1 = x_1x_4$
3. $y_2x_0 = x_0y_3$
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6. $x_2z_1 = z_1x_4$
7. $y_1z_0 = z_0y_3$
8. $y_2z_1 = z_1y_4$
9. $x_0z_1z_0 = z_0x_2x_1x_0$
10. $x_1z_2z_1 = z_1x_3x_2x_1$

where $x_3 = x_1^{-1}x_2x_1$, $x_4 = x_2^{-1}x_3x_2$, $y_3 = x_1^{-1}y_2x_1$, $y_4 = x_2^{-1}y_3x_2$, $z_0 = y_1^{-1}y_0x_1x_0$, $z_1 = y_2^{-1}y_1x_2x_1$, and $z_2 = y_3^{-1}y_2x_3x_2$.

The Metric

Using the normal form, we can obtain upper and lower bounds on the metric of $F(n_1, ..., n_k)$ in terms of the number of leaves present in the minimal tree-pair diagram representative.

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Theorem

For a given element $w \in F(n_1, ..., n_k)$, where $L(w)$ denotes the number of leaves in the minimal tree-pair diagram representative of w, there exist fixed c_1, c_2, c_3, c_4 such that

$$c_1 log L(w) + c_2 \leq |w|_{\{x_0, x_1, y_0, y_1\}} \leq c_3 L(w) + c_4$$

Both of these bounds are sharp.

The Metric

An example of an element with length of logarithmic order with respect to the number of leaves in its minimal tree-pair diagram representative:

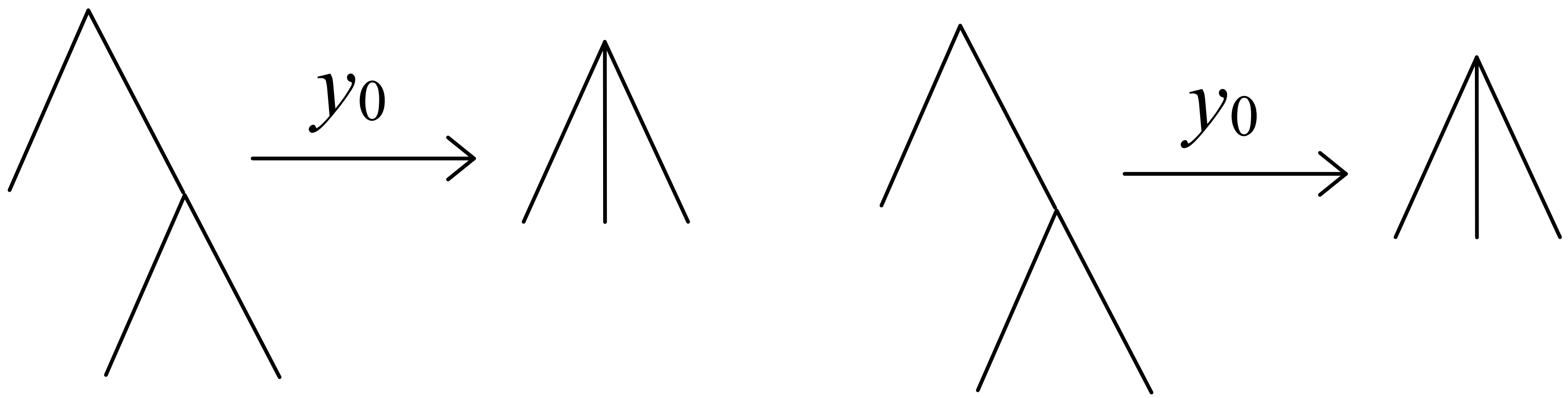

Figure: Computing y_0^n in $F(2,3)$

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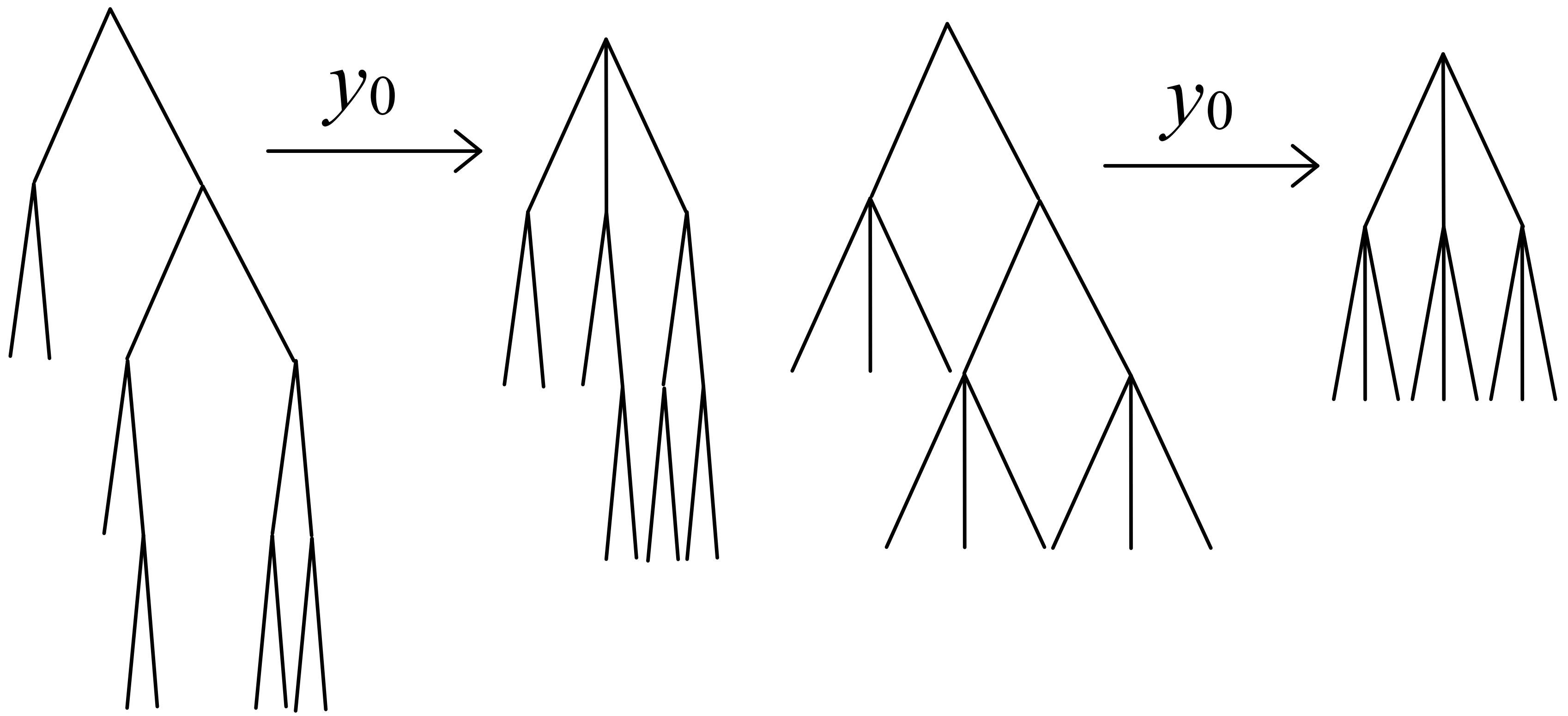

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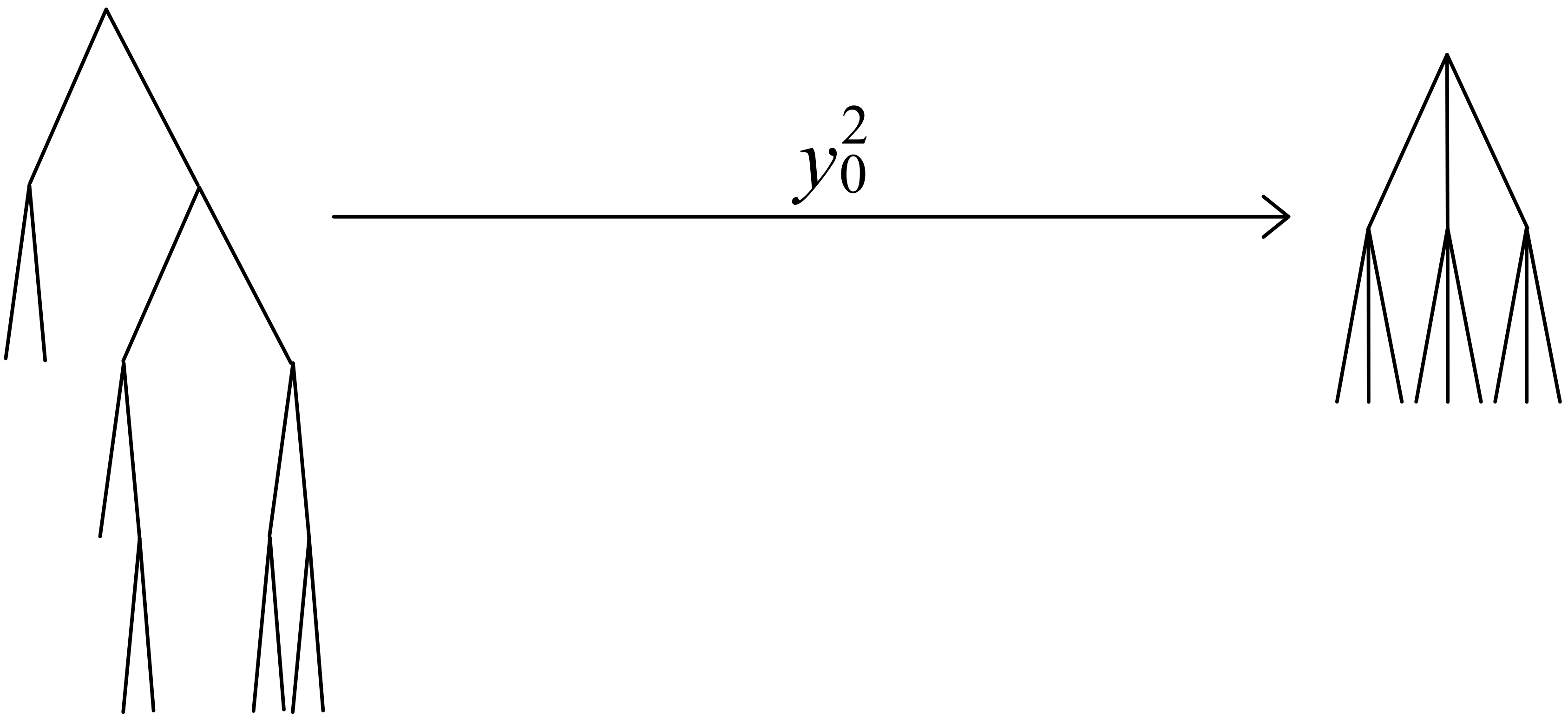

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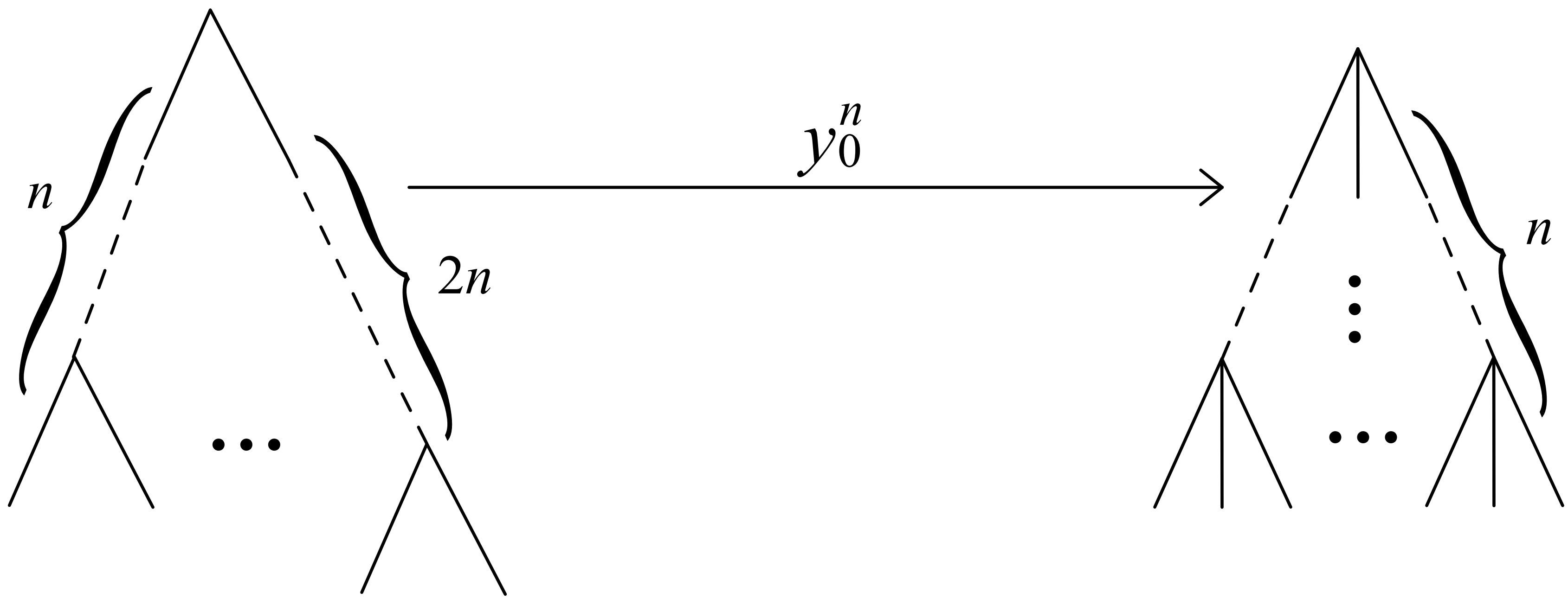
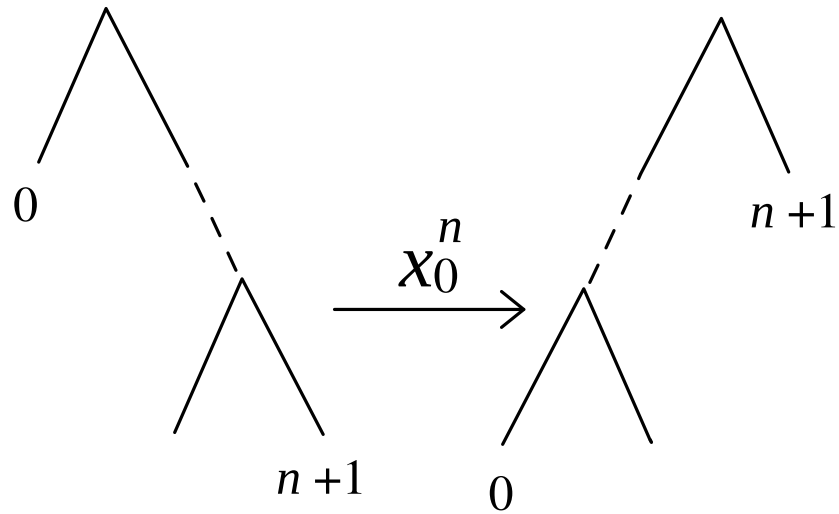


Figure: Computing y_0^n in $F(2,3)$

$L(y_0^n) = 3^n$, but $|y_0^n|_{\{x_0,x_1,y_0,y_1\}} \leq n$.
So the lower bound on our metric estimate is sharp.

The Metric

An example of an element with length of linear order with respect to the number of leaves in its minimal tree-pair diagram representative:



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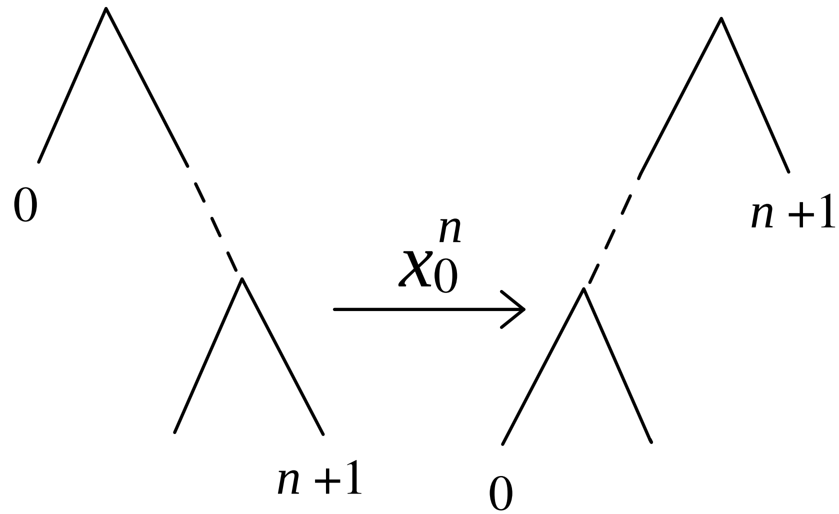

Lemma

If $D(w)$ stands for the depth of w (i.e. the maximum length from the root vertex to any leaf vertex in the minimal tree-pair diagram representative), then $|w|_{\{x_0,x_1,y_0,y_1\}} \geq \frac{D(w)}{3}$.

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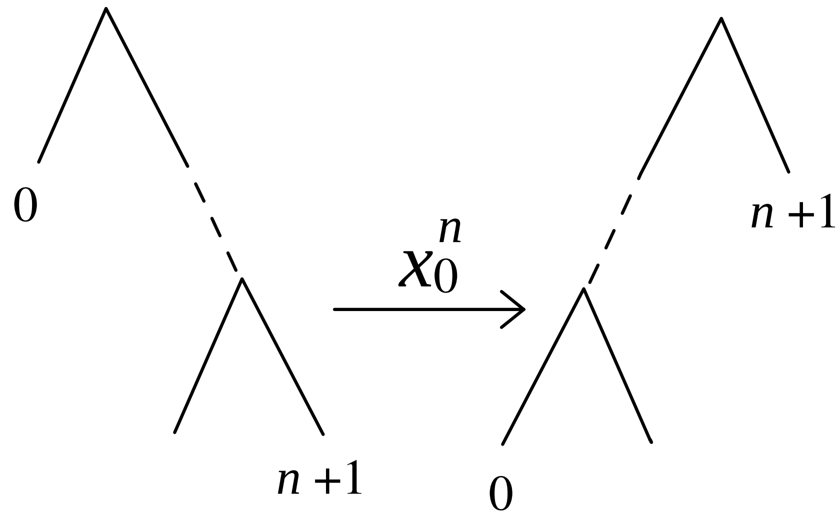

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$L(x_0^n) = D(x_0^n) + 1$, so $\frac{L(x_0^n)-1}{3} \leq |x_0^n|_{\{x_0,x_1,y_0,y_1\}} \leq c_1 L(x_0^n) + c_2$.
So the upper bound on our metric estimate is sharp.

Subgroup Embeddings

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Theorem

For any n_i such that there exists n_j with $j \in \{1, ..., k\}$, $i \neq k$, and $n_i - 1 | n_j - 1$, $F(n_i)$ is exponentially distorted in $F(n_1, ..., n_k)$.

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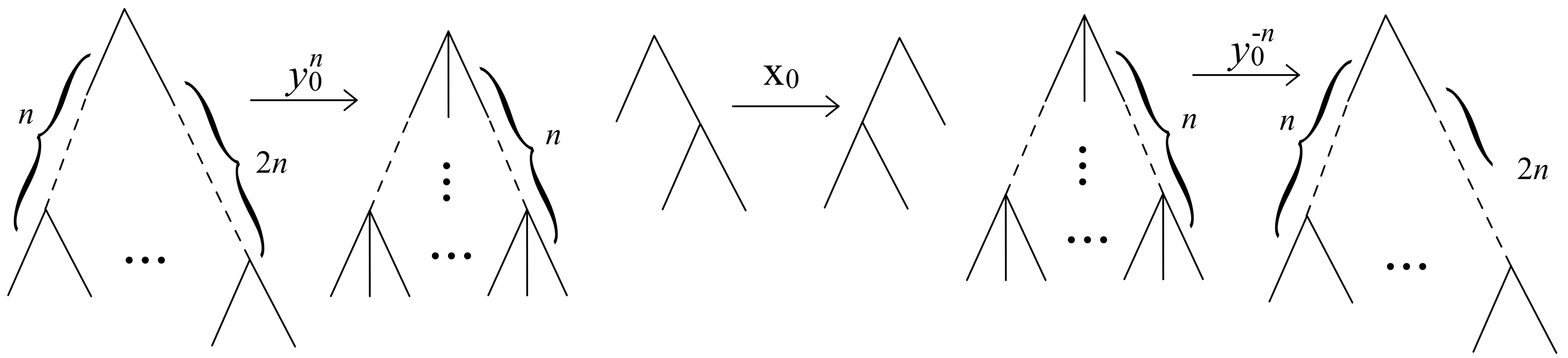

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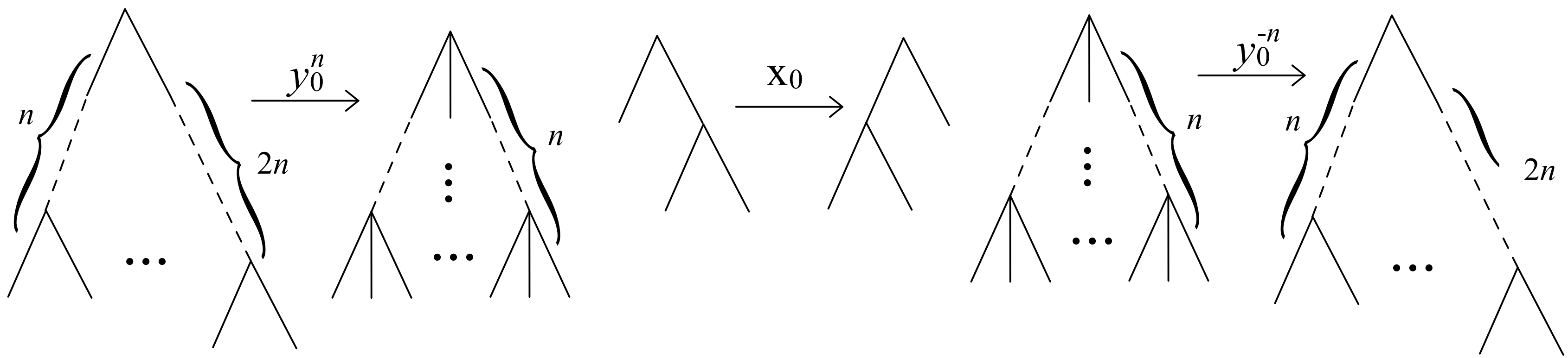

Let $w_n = y_0^{-n} x_0 y_x^n$. $L(w_n)$ is of the order 3^n, so $|w_n|_{\{x_0, x_1\} \in F(2)}$ is of the order 3^n. But $|w_n|_{\{x_0, x_1, y_0, y_1\} \in F(2,3)} \leq 2n + 1$.
So $|w_n|_{\{x_0, x_1\} \in F(2)}$ grows exponentially with respect to $|w_n|_{\{x_0, x_1, y_0, y_1\} \in F(2,3)}$.

Subgroup Embeddings - Future Questions

- Does $F(n_i)$ quasi-isomorphically embed into $F(n_1, ..., n_k)$ when there exists no other n_j such that $n_i - 1 | n_j - 1$?

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- Do cyclic subgroups of $F(n_1, ..., n_k)$ embed quasi-isometrically?